On almost perfect linear Lee codes of minimum distance 5

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Abstract. More than 50 years ago, Golomb and Welch conjectured that there is no perfect Lee codes C of minimum distance $2r+1$ in $\mathbb{Z}^n$ for $r \geq 2$ and $n \geq 3$. Recently, Leung and the second author proved that if C is linear, then the Golomb-Welch conjecture is valid for $r = 2$ and $n \geq 3$. In this paper, we consider the classification of linear Lee codes with the second best possibility, that is the density of the lattice packing of $\mathbb{Z}^n$ by Lee spheres $S(n,r)$ equals $\frac{|S(n,r)|}{|S(n,r)|+1}$. We show that, for $r = 2$ and $n \equiv 0, 3, 4 \pmod 6$, this packing density can never be achieved.

Keywords: Lee metric · perfect code · Golomb-Wech conjecture · Cayley graph

1 Introduction

Let $\mathbb{Z}$ denote the ring of integers. For two words $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n) \in \mathbb{Z}^n$, the *Lee distance* (also known as ℓ_1-norm, taxicab metric, rectilinear distance or Manhattan distance) between them is defined by

$$d_L(x, y) = \sum_{i=1}^{n} |x_i - y_i| \text{ for } x, y \in \mathbb{Z}^n.$$

A *Lee code* C is just a subset of $\mathbb{Z}^n$ endowed with the Lee distance. If C further has the structure of an additive group, i.e. C is a lattice in $\mathbb{Z}^n$, then we call C a *linear Lee code*. Lee codes have many practical applications, for example, constrained and partial-response channels [18], flash memory [19] and interleaving schemes [4].

The minimum distance between any two distinct elements in C is called the *minimum distance* of C. Given a Lee code of minimum distance $2r+1$, for any $x \in \mathbb{Z}^n$, if there is always a unique $c \in C$ such that $d_L(x, c) \leq r$, then $\mathcal{C}$ is called a *perfect code*. This is equivalent to

$$\mathbb{Z}^n = \dot{\bigcup}_{c \in C} (S(n,r) + c),$$

where $S(n,r) := \{(x_1, \cdots, x_n \in \mathbb{Z}^n : \sum_{i=1}^{n} |x_i| \leq r\}$ and $S(n,r) + c := \{v + c : v \in S(n,r)\}$. Thus, the existence of a perfect Lee code implies a tiling of $\mathbb{Z}^n$ by Lee spheres of radius r.

Perfect Lee codes exist for $n = 1, 2$ and any r, and for $n \geq 3$ and $r = 1$. Golomb and Welch [6] conjectured that there are no more perfect Lee codes for other choices of n and r. This conjecture is still far from being solved, despite many efforts and various approaches applied on it. We refer the reader to the recent survey [9] and the references therein.

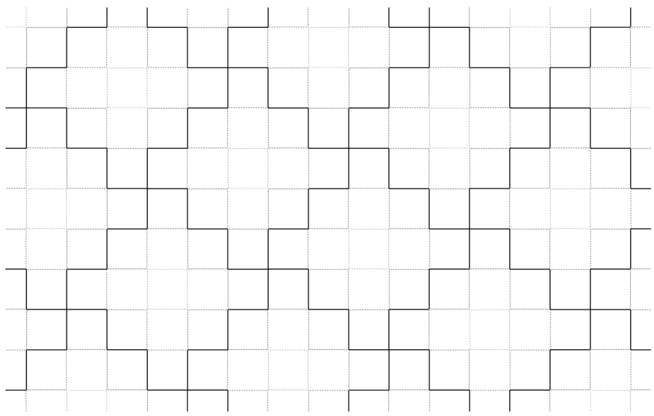

Fig. 1. Tiling of $\mathbb{R}^2$ by $L(2, 2)$

Let $L(n, r)$ denote the union of n-cubes centered at each point of $S(n, r)$ in $\mathbb{R}^n$. It is not difficult to see that there is a tiling of $\mathbb{Z}^n$ by $S(n, r)$ if and only if there is a tiling of $\mathbb{R}^n$ by $L(n, r)$. Figure 1 shows a (lattice) tiling of $\mathbb{R}^2$ by $L(2, 2)$. As the shape of $L(n, r)$ is close to a cross-polytope when r is large enough, one can use the cross-polytope packing density to prove the Golomb-Welch conjecture provided that r is large enough compared with n. In fact, this idea was first applied by Golomb and Welch themselves in [6]. There are some other geometric approaches, including the analysis of some local configurations of the boundary of Lee spheres in a tiling by Post [15], and the density trick by Astola [2] and Lepistö [11].

However, it seems that the geometric approaches do not work for small r and large n. In the past several years, algebraic approaches have been proposed and applied on the existence of perfect linear Lee code for small r. In [10], Kim introduced a symmetric polynomial method to study this problem for sphere radius r = 2. This approach has been extended by Zhang and Ge [20], and Qureshi [16] for $r \geq 3$. See [9, 17, 21] for other related results. In particular, Leung and the second author [12] succeeded in getting a complete solution to the case with $r = 2$: there is no perfect linear Lee code of minimum distance 5 in $\mathbb{Z}^n$ for $n \geq 3$.

It is worth pointing out that the existence of a perfect linear Lee code of minimum distance $2r + 1$ in $\mathbb{Z}^n$ is equivalent to an abelian Cayley graph of degree $2n$ and diameter r whose number of vertices meets the so-called *abelian*

Cayley Moore bound; see [5, 21]. For more results about the degree-diameter problems in graph theory, we refer to the survey [14].

The *packing radius* of a Lee code C is defined to be the largest integer r' such that for any element $w \in \mathbb{Z}^n$ there exists at most one codeword $c \in C$ with $d_L(w, c) \le r'$. For a Lee code C of packing radius r, let S_c denote the Lee sphere of radius r centered at the codeword $c \in C$. The packing density of C is defined to be $\lim_{\ell\to\infty} \frac{\sum_{c\in C} |S(n,\ell)\cap S_c|}{|S(n,\ell)|}$ if the limit exists.

It is clear that a perfect Lee code C means the packing density of $\mathbb{Z}^n$ by $S(n, r)$ with centers consisting of all the elements in C is 1. As there is no perfect linear Lee code known for $r \ge 2$ and $n \ge 3$, one may wonder whether the second best is possible, which is about the existence of a lattice packing of $\mathbb{Z}^n$ by $S(n, r)$ with density $\frac{\#S(n,r)}{\#S(n,r)+1}$. We call such a linear Lee code *almost perfect*. In Figure 2, we present a lattice packing of $\mathbb{Z}^2$ by $S(2, 2)$, and its packing density is $\frac{\#S(2,2)}{\#S(2,2)+1} = \frac{13}{14}$. This example and the numbers labeled on the cubes will be explained later in Example 1. For convenience, we abbreviate the term almost perfect linear Lee code to *APLL* code.

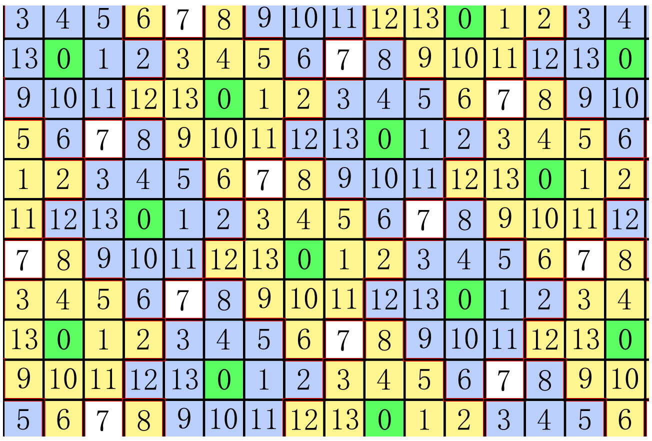

Fig. 2. An almost perfect Lee code in $\mathbb{Z}^2$

Note that an APLL code also satisfies the definition of a quasi-perfect Lee code introduced in [1], but a quasi-perfect Lee code is not necessarily almost perfect; see [5, 13] for constructions and other results on quasi-perfect Lee codes.

About the existence of APLL codes of minimum distance 5, one can apply the symmetric polynomial method in [10] and the algebraic number theory approach in [21] to derive some partial results. For $n \le 10^5$, one can exclude the existence of APLL codes of minimum distance 5 for 76,573 choices of n. For more details, see [7].

The main result of this paper is the following one.

Theorem 1. *Let n be a positive integer larger than 2. If $n \equiv 0, 3, 4 \pmod 6$, then there exists no almost perfect linear Lee code of minimum distance 5.*

As in [12], our proof is given in the group ring $\mathbb{Z}[G]$ where G is of order $|S(n,2)|+1 = 2(n^2+n+1)$. However, the situation here is different from the one appeared in [12]. We consider a special subset $T \subseteq G$ and show that T splits into two disjoint subsets $T_0 \in H$ and $fT_1 \in fH$ where H is the unique subgroup of G of index 2 and f is the unique involution in G. Then, we analyze the elements appearing in $(T_0^{(2)} + T_1^{(2)})T_0$ and $(T_0^{(2)} + T_1^{(2)})T_1$. Unfortunately, we cannot get any contradiction when $n \equiv 1, 2, 5 \pmod 6$. But we conjecture that there is no APLL code for this case when $n > 2$.

The rest part of this paper consists of two sections. In Section 2, we convert the existence of an almost perfect linear Lee code of radius 2 into some conditions in group ring. Then we prove Theorem 1 in Section 3.

2 A necessary and sufficient condition in group ring

The following result converts the existence of APLL codes into an algebraic combinatorics problem on abelian groups. Its proof is not difficult, and more or less the same as the proof of Theorem 6 in [8]. Hence we omit its proof here.

Lemma 1. *There is an APLL code of minimum distance $2r+1$ in $\mathbb{Z}^n$ if and only if there is an abelian group G and a homomorphism $\varphi : \mathbb{Z}^n \to G$ such that the restriction of φ to $S(n,r)$ is injective and $G \setminus \varphi(S(n,r))$ has only one element.*

The next result translates the existence of an APLL code into a group ring condition. The same result has been proved in [7] in the context of the existence of abelian Moore Cayley graphs with excess one. We refer to [3, Chapter VI, §3] for the definition and basic properties of group rings.

Lemma 2. *There exists an APLL code of radius 2 in $\mathbb{Z}^n$ if and only if there is an abelian group G of order $2(n^2+n+1)$ and an inverse-closed subset $T \subseteq G$ containing e with $|T| = 2n+1$ such that*

$$T^2 = 2(G - f) - T^{(2)} + 2ne, \tag{1}$$

where e is the identity element and f is the unique element of order 2 in G.

The following two examples show that for $n = 1, 2$, there do exist APLL codes of radius 2. The corresponding result of Example 1 (b) has been already depicted in Figure 2.

Example 1. Let G be a cyclic group generated by g of order $2n^2 + 2n + 2$.

(a) $n = 1$, $|G| = 6$ and $T = \{e, g^{\pm 1}\}$.
(b) $n = 2$, $|G| = 14$ and $T = \{e, g^{\pm 1}, g^{\pm 4}\}$.

In Figure 2, we label each element in $\mathbb{Z}^2$ which is mapped to $\varphi(e_1) = g$ by 1 and those mapped to $\varphi(e_2) = g^4$ by 4. The center of every Lee sphere is labeled by 0. The holes are all labeled by 7 corresponding to the unique involution g^7 in G.

Let H denote the unique subgroup of order n^2+n+1 in G. Define $T_0 = T \cap H$ and $T_1 = fT \cap H$. Thus $T = T_0 + fT_1$. By (1),

$$T_0^2 + T_1^2 + 2fT_0T_1 = 2(G - f) - T_0^{(2)} - T_1^{(2)} + 2n \cdot e.$$

The left-hand side and the right-hand side of the above equation can both be viewed as multisets in G. Hence we may concentrate on the elements in H and fH, respectively, to derive

$$T_0T_1 = H - e, \tag{2}$$

$$T_0^2 + T_1^2 = 2H - T_0^{(2)} - T_1^{(2)} + 2ne. \tag{3}$$

Therefore, we have proved the following result.

Lemma 3. *Let $T = T_0 + fT_1 \subseteq G$ with T_0 and $T_1 \subseteq H$. The subset T satisfies that $e \in T$, $T^{(-1)} = T$ and* (1) *if and only if $e \in T_0$, $T_0^{(-1)} = T_0$, $T_1^{(-1)} = T_1$,* (2) *and* (3) *hold.*

The following result is a collection of obvious necessary conditions for T_0 and T_1, which will be intensively used in Section 3. We omit the proof.

Lemma 4. *Suppose that $T = T_0 + fT_1 \subseteq G$ satisfying $e \in T$, $T^{(-1)} = T$, $|T| = 2n + 1$,* (2) *and* (3)*. Then the following statements hold.*

(a) $e \in T_0$, $e \notin T_1$;
(b) $T_0 \cap T_1 = \emptyset$ and $T_0^{(2)} \cap T_1^{(2)} = \emptyset$;
(c) $T_0 \cap (T_0^{(2)} \setminus \{e\}) = T_0 \cap T_1^{(2)} = \emptyset$;
(d) $\{ab : a \neq b, a, b \in T_0\} \cap T_0^{(2)} = \{e\}$;
(e) When n is odd, $|T_0| = n$ and $|T_1| = n + 1$;
(f) When n is even, $|T_0| = n + 1$ and $|T_1| = n$.
(g) There is no common non-identity element in T_0^2 and T_1^2.
(h) $T_0 \cap T_0^{(3)} = \{e\}$.

3 Proof of the main result

In this section, we are going to prove Theorem 1 by showing the nonexistence of inverse-closed subsets $T_0, T_1 \subseteq H$ satisfying $e \in T_0$, $|T_0| + |T_1| = 2n + 1$, (2) and (3).

Define $\hat{T} = T_0 + T_1 \in \mathbb{Z}[H]$. Write $\hat{T}^{(2)} = \sum_{i=0}^{2n} a_i$ where $a_0 = e$, $a_i \in T_0^{(2)}$ for $i = 0, \cdots, |T_0| - 1$ and $a_i \in T_1^{(2)}$ for $i = |T_0|, \cdots, 2n$. Let $k_0 = |T_0|$ and $k_1 = |T_1|$.

By multiplying T_0 and T_1 on both sides of (3), respectively, and rearranging the terms, we get

$$\hat{T}^{(2)}T_0 = (2k_0 - k_1)H + T_1 - T_0^3 + 2nT_0,$$
$$\hat{T}^{(2)}T_1 = (2k_1 - k_0)H + T_0 - T_1^3 + 2nT_1.$$

Consider the above two equations modulo 3

$$\hat{T}^{(2)}T_0 \equiv (2k_0 - k_1)H + T_1 - T_0^{(3)} + 2nT_0 \pmod 3, \tag{4}$$
$$\hat{T}^{(2)}T_1 \equiv (2k_1 - k_0)H + T_0 - T_1^{(3)} + 2nT_1 \pmod 3. \tag{5}$$

Noting that $T_0^3 \equiv T_0^{(3)} \pmod 3$ and $T_0^{(3)}$ is very close to a subset in H. In fact, if $3 \nmid |H|$, then $T_0^{(3)}$ is a subset; for $3 \mid |H|$, we will handle it in Lemma 6. Thus, we will know the coefficients modulo 3 of most of the elements in the right-hand side of the above two equations. For instance, when $n \equiv 1 \pmod 3$ and n is odd, the first equation becomes $\hat{T}^{(2)}T_0 \equiv T_1 - T_0^{(3)} + 2T_0 \pmod 3$. As T_1, $T_0^{(3)}$ and T_0 are approximately of size n, most of the elements in H appear in $\hat{T}^{(2)}T_0$ for $3k$ times, $k = 0, 1, \cdots$.

Let X_i (Y_i, resp.) be the subset of elements of H appearing in $\hat{T}^{(2)}T_0$ ($\hat{T}^{(2)}T_1$, resp.) exactly i times for $i = 0, 1, \cdots, M_0$ (M_1, resp.), which means X_i's (Y_i's, resp.) form a partition of the group H. In particular, we define M_0 and M_1 such that X_{M_0} and Y_{M_1} are non-empty sets. Then

$$\hat{T}^{(2)}T_0 = \sum_{i=0}^{M_0} iX_i, \quad \hat{T}^{(2)}T_1 = \sum_{i=0}^{M_1} iY_i.$$

By the above equations, we get three conditions on the value of $|X_i|$'s and $|Y_i|$'s:

$$\sum_{i=1}^{M_0} i|X_i| = (2n+1)k_0, \tag{6}$$
$$\sum_{i=1}^{M_1} i|Y_i| = (2n+1)k_1, \tag{7}$$
$$\sum_{i=0}^{M_0} |X_i| = \sum_{i=0}^{M_1} |Y_i| = n^2 + n + 1. \tag{8}$$

Some extra conditions are given by the following Lemma.

Lemma 5. *Let* $\theta_0 = |(T_0^2 \setminus T_0^{(2)}) \cap \hat{T}^{(4)}|$ *and* $\theta_1 = \frac{|T_1 \cap \hat{T}^{(2)}|}{2} + |(T_1^2 \setminus (T_1^{(2)} \cup \{e\})) \cap \hat{T}^{(4)}|$. *Then*

$$\sum_{i=1}^{M_0} |X_i| = (2n+1)k_0 - 2(k_0 - 1)k_0 + \theta_0 + \sum_{s \geq 3} \frac{(s-1)(s-2)}{2}|X_s|, \tag{9}$$
$$\sum_{i=1}^{M_1} |Y_i| = (2n+1)k_1 - 2(k_1 - 1)k_1 + \theta_1 + \sum_{s \geq 3} \frac{(s-1)(s-2)}{2}|Y_s|. \tag{10}$$

Notice that $|X_i|$'s and $|Y_i|$'s are nonnegative integers. Our main idea is to use (6) , (7) , (8), (9) and (10) together with $\hat{T}^{(2)}T_0$ and $\hat{T}^{(2)}T_1 \pmod 3$ to determine $|X_i|$'s and $|Y_i|$'s. For some cases, we will end up with a contradiction which means there is no T_0 and T_1 satisfying (2) and (3), for example, see Theorem 2; for other cases, the value of $|X_i|$'s and $|Y_i|$'s together with a careful analysis of (2) and (3) also lead to contradictions, for instance, see Theorems 4.

To prove Lemma 5, we need to prove a series of lemmas which will appear in the full version of this paper.

In the following, we investigate (4) and (5) separately in different cases depending on the value of n modulo 3. For $n \equiv 1 \pmod 3$, we first need the following observation.

Lemma 6. *When $n \equiv 1 \pmod 3$,*

(a) there is no element in $T_j^{(3)}$ appearing more than 2 times for $j = 0, 1$,
(b) e appears only once in $T_0^{(3)}$, and
(c) there are 0 or 2 elements appearing twice in $T_0^{(3)}$, and there are at most 2 elements appearing twice in $T_1^{(3)}$.

By Lemma 4 (e) and (f), $2k_0 - k_1 \equiv 2k_1 - k_0 \equiv 0 \pmod 3$. Using (4), (5) and $3 \mid (n-1)$,

$$\hat{T}^{(2)}T_0 \equiv T_1 - T_0^{(3)} + 2T_0 \pmod 3, \tag{11}$$

$$\hat{T}^{(2)}T_1 \equiv T_0 - T_1^{(3)} + 2T_1 \pmod 3. \tag{12}$$

By Lemma 4 (b) and (h), $T_0 \cap T_1 = \emptyset$ and $T_0 \cap T_0^{(3)} = \{e\}$. Let Δ_i be the set of elements appearing twice in $T_i^{(3)}$ for $i = 1, 2$. By Lemma 6, $|\Delta_0|, |\Delta_1| \le 2$ and there is no element appearing more than 2 times in $T_i^{(3)}$ for $i = 1, 2$. Notice that

$$|T_0^{(3)}| = k_0 - |\Delta_0|, \text{ and } |T_1^{(3)}| = k_1 - |\Delta_1|,$$

where $|T_i^{(3)}|$ denotes the number of distinct elements in the multiset $T_i^{(3)}$ for $i = 1, 2$. Depending on the parity of n, we investigate (12) and (11), respectively.

Theorem 2. *For any n satisfying n odd, $n \equiv 1 \pmod 3$ and $n > 2$, there is no inverse-closed subsets T_0 and $T_1 \subseteq H$ with $e \in T_0$ and $k_0 + k_1 = 2n+1$ satisfying (2) and (3).*

Proof. As $2 \mid n$, $k_0 = n+1$ and $k_1 = n$. We concentrate on (12). Let $u_0 = |T_1^{(3)} \cap T_0|$ and $u_1 = |T_1^{(3)} \cap T_1|$. By Lemma 4, $T_0 \cap T_1 = \emptyset$. Furthermore, we separate Δ_1 into three disjoint parts $\Delta_1^0 = \Delta_1 \cap T_0$, $\Delta_1^1 = \Delta_1 \cap T_1$ and $\Delta_1^2 = \Delta_1 \setminus (T_0 \cup T_1)$.

By comparing the coefficients of elements in (12), we get

$$\bigcup_{i \ge 0} Y_{3i+2} = (T_1 \setminus T_1^{(3)}) \,\dot{\cup}\, \left(T_1^{(3)} \setminus (T_0 \,\dot{\cup}\, T_1 \cup \Delta_1)\right) \,\dot{\cup}\, \Delta_1^0.$$

Notice that $|T_1^{(3)}| = k_1 - |\varDelta_1|$. It follows that

$$\sum_{i\geq 0} |Y_{3i+2}| = (n-u_1) + (n - |\varDelta_1| - (u_0+u_1) - |\varDelta_1^2|) + |\varDelta_1^0|.$$

Similarly,

$$\bigcup_{i\geq 0} Y_{3i+1} = (T_0 \setminus T_1^{(3)}) \mathbin{\dot\cup} \varDelta_1^2 \mathbin{\dot\cup} (T_1 \cap T_1^{(3)} \setminus \varDelta_1^1).$$

Thus

$$\sum_{i\geq 0} |Y_{3i+1}| = (n+1-u_0) + |\varDelta_1^2| + u_1 - |\varDelta_1^1|.$$

To summarize, we have proved

$$\sum_{i\geq 0} |Y_{3i+2}| = 2n - 2u_1 - u_0 - 2|\varDelta_1^2| - |\varDelta_1^1|, \ \sum_{i\geq 0} |Y_{3i+1}| = n+1-u_0+u_1+|\varDelta_1^2|-|\varDelta_1^1|, \tag{13}$$

Now (10) becomes

$$\begin{aligned}\sum_{i=1}^{M_1} |Y_i| &= (2n+1)n - 2(n-1)n + \theta_1 + \sum_{s\geq 3} \frac{(s-1)(s-2)}{2}|Y_s| \\ &= 3n + \theta_1 + \sum_{s\geq 3} \frac{(s-1)(s-2)}{2}|Y_s|. \end{aligned} \tag{14}$$

Plugging the two equations in (13) into it to replace $3n$, we get

$$\begin{aligned}\sum_{i=1}^{M_1} |Y_i| =& \sum_{i\geq 0} |Y_{3i+1}| + \sum_{i\geq 0} |Y_{3i+2}| + 2u_0 + u_1 + |\varDelta_1^2| + 2|\varDelta_1^1| - 1 \\ &+ \theta_1 + \sum_{s\geq 3} \frac{(s-1)(s-2)}{2}|Y_s|,\end{aligned}$$

which implies

$$\begin{aligned}1 =& 2u_0 + u_1 + |\varDelta_1^2| + 2|\varDelta_1^1| + \theta_1 \\ &+ \sum_{s>3, 3|s} \left(\frac{(s-1)(s-2)}{2} - 1\right)|Y_s| + \sum_{s\geq 3, 3\nmid s} \left(\frac{(s-1)(s-2)}{2}\right)|Y_s|.\end{aligned} \tag{15}$$

As $|Y_i|$, u_0, u_1,$|\varDelta_1^2|$, $|\varDelta_1^1|$ and θ are all nonnegative integers, (15) implies that $u_0 = 0$ and each $|Y_i| = 0$ for $i \geq 4$. Recall that u_1 is the cardinality of the inverse-closed subset $T_1^{(3)} \cap T_1$ which does not contain $e \in H$. Hence u_1 must be even. Similarly, $|\varDelta_1^1|$ and $|\varDelta_1^2|$ are also even. Thus, by (15),

$$u_1 = |\varDelta_1^1| = |\varDelta_1^2| = 0,$$

and $\theta_1 = 1$. Plugging them into (13), we get

$$|Y_1| = n + 1, \ |Y_2| = 2n.$$

By (7),

$$3|Y_3| = (2n + 1)n - (n + 1) - 4n = 2n^2 - 4n - 1.$$

As Y_i's form a partition of the group H and each Y_i is inverse-closde, there exists only one of them of odd size. However, now $|Y_1|$ and $|Y_3|$ are both odd, which is impossible.

Therefore, we have excluded the existence of $|Y_i|$'s which means there do not exist inverse-closed subsets T_0 and $T_1 \subseteq H$ with $e \in T_0$ and $k_0 + k_1 = 2n + 1$ satisfying (2) and (3). □

For n odd and $n \equiv 1 \pmod 3$, we cannot derive any contradiction by the value of $|X_i|$'s and $|Y_i|$'s. Instead, we can completely determine the value of $|X_i|$'s and we omit its proof.

Theorem 3. *For $n \equiv 1 \pmod 6$, if there exist inverse-closed subsets T_0 and $T_1 \subseteq H$ with $e \in T_0$ and $k_0 + k_1 = 2n + 1$ satisfying (2) and (3), then*

$$\begin{aligned} |X_0| &= \frac{1}{3}(n-1)^2, |X_1| = n + 2, |X_2| = 2n - 2, \\ |X_3| &= \frac{2}{3}(n-1)^2, |X_i| = 0, \text{ for } i > 3. \end{aligned}$$

Furthermore, $T_1 \cap T_0^{(3)} = \emptyset$ and $\Delta_0 \subseteq T_1$.

The proof for $n \equiv 0 \pmod 3$ needs different approach and it is more complicated. We will provide their detailed proof (11 pages) in the full version of this paper.

Theorem 4. *For positive integer n satisfying $3 \mid n$, there is no inverse-closed subsets T_0 and $T_1 \subseteq H$ with $e \in T_0$ and $|T_0| + |T_1| = 2n + 1$ satisfying (2) and (3).*

The main result, i.e. Theorem 1 is a simple combination of Theorems 2 and 4. For $n \equiv 2 \pmod 3$, a similar analysis of (4) and (5) can only tell us the ranges of the value of $|X_1|$ and $|X_4|$ which leads to no contradiction.

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