Antipodal two-weight rank-metric codes*

Rakhi Pratihar[1] and Tovohery Hajatiana Randrianarisoa[2]

[1] University of Tromsø The Arctic University of Norway, Norway
pratihar.rakhi@gmail.com
[2] Florida Atlantic University, Boca Raton, Florida, USA
tovo@aims.ac.za

Abstract. We consider the class of linear antipodal two-weight rank-metric codes and discuss their properties and characterization in terms of t-spreads. It is shown that the dimension of such codes is 2 and the minimum rank distance is at least half of the length. We construct antipodal two-weight rank-metric codes from certain MRD codes. A complete classification of such codes is obtained, when the minimum rank distance is equal to half of the length.

Keywords: Rank-metric codes · Antipodal two-weight · t-Spreads · Subspreads

1 Introduction

Let q be a prime power and let $\mathbb{F}_{q^m}$ be the field extension of degree m over the finite field $\mathbb{F}_q$. For a positive integer n, the *rank* of an element $\mathbf{c} = (c_1, \ldots, c_n)$ in $\mathbb{F}_{q^m}^n$ is defined by $\mathrm{rank}(\mathbf{c}) = \dim_{\mathbb{F}_q} \langle c_1, \ldots, c_n \rangle_{\mathbb{F}_q}$, where $\langle c_1, \ldots, c_n \rangle_{\mathbb{F}_q}$ is the $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^m}$ generated by the c_i's. The function "rank" induces a metric d_r on $\mathbb{F}_{q^m}^n$ where $\mathrm{d}_r(\mathbf{c}, \mathbf{c}') = \mathrm{rank}(\mathbf{c} - \mathbf{c}')$ for $\mathbf{c}, \mathbf{c}'$ in $\mathbb{F}_{q^m}^n$. An $[n, k, d]$ (linear) rank-metric code $\mathcal{C}$ over $\mathbb{F}_{q^m}/\mathbb{F}_q$ is an $\mathbb{F}_{q^m}$-linear subspace of $\mathbb{F}_{q^m}^n$ of dimension k such that $\min_{\mathbf{c} \in \mathcal{C} \setminus \{\mathbf{0}\}} \mathrm{rank}(\mathbf{c}) = d$. An $[n, k, d]$ linear rank-metric code over $\mathbb{F}_{q^m}/\mathbb{F}_q$ is called an *antipodal two-weight* or ATW if $d \neq n$, any nonzero codeword has rank either d or n and there is at least one codeword of full rank n.

Linear codes with few distinct weights are important in coding theory, both from practical and theoretical point of view. For codes with Hamming metric, the *constant weight codes* i.e. codes where all the nonzero codewords have same weight and the *two-weight* linear codes have been studied extensively. In [3], Bonisoli gives a characterization of the constant weight codes. The class of two-weight linear Hamming metric codes has been investigated by Delsarte [5]; see [4] for a systematic exposition. The classification of the subclass of ATW linear codes with Hamming metric are obtained in [7].

We are interested in the q-analogues of some of these results. For linear rank-metric codes, the class of constant rank weight codes are completely classified

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in [9] following a geometric approach. It is proved that, up to equivalence, there is only one non-degenerate constant rank weight code if the code has dimension at least 2. In this article we consider the class of ATW rank-metric codes. Using the classification of constant rank weight codes, we show that the dimension of such codes is 2 and the minimum distance d must be at least $\frac{n}{2}$. We provide an equivalent characterization of ATW rank-metric codes in terms of t-spreads, more precisely, as t-subspreads of Desarguesian t-spreads induced by q-systems associated to the codes. It is proved that, up to equivalence, there exists only one ATW code for the case $d = \frac{n}{2}$ and a complete classification of such codes is also obtained. For the case of $d > \frac{n}{2}$, we get partial results regarding the classification. We construct ATW rank-metric codes using MRD codes of suitable parameters.

The article is organized as follows. In the next section, we collect some preliminaries about notions such as rank-metric codes and t-spreads. In Section 3, we derive some properties of the ATW rank-metric codes and t-spreads and establish a relation between these two objects. Section 4 deals with classification and construction of ATW rank-metric codes.

2 Preliminaries

Throughout we use $\mathbf{a} \cdot \mathbf{b}$ to denote the usual dot product of two vectors $\mathbf{a}, \mathbf{b}$.

2.1 Rank-metric codes

In [5], it was shown that for any $[n, k, d]$-rank metric code over $\mathbb{F}_{q^m}/\mathbb{F}_q$, $d \leq n - k + 1$. If $d = n - k + 1$, then the code is called a maximum rank distance (MRD) code.

Following the definition of [8], two rank-metric codes $\mathcal{C}$ and $\mathcal{C}'$ over $\mathbb{F}_{q^m}/\mathbb{F}_q$ are called equivalent if $\mathcal{C} = \alpha\mathcal{C}'\mathbf{M}$ for some $\alpha \in \mathbb{F}_{q^m}^{\times}$ and $\mathbf{M} \in \mathbb{F}_q^{n\times n}$ invertible, where $\mathcal{C}'\mathbf{M} = \{\mathbf{c}\mathbf{M} \colon \mathbf{c} \in \mathcal{C}'\}$. It is equivalent to saying that the corresponding generator matrices $\mathbf{G}$ and $\mathbf{G}'$ satisfy $\mathbf{G} = \mathbf{G}'\mathbf{M}$.

A rank-metric code $\mathcal{C}$ is *non-degenerate* if the columns of its generator matrix $\mathbf{G}$ are linearly independent over $\mathbb{F}_q$. Otherwise, if $\mathcal{C}$ is degenerate, then $\mathcal{C}$ is equivalent to a code $\{(\mathbf{c}|\mathbf{0}) \colon \mathbf{c} \in \mathcal{C}'\}$ where $\mathcal{C}'$ is non-degenerate. Therefore throughout this paper, our code is assumed to be non-degenerate unless otherwise specified.

In [9], rank-metric codes are described in terms of called q-systems.

Definition 1 (q-system). *Let k, n be positive integers such that $k \leq n$. An $[n, k]$ q-system X in $\mathbb{F}_{q^m}^k$ is an $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^m}^k$ of dimension n. If $\{\mathbf{c}_i \in \mathbb{F}_{q^m}^k : 1 \leq i \leq n\}$ are an $\mathbb{F}_q$-basis of X, then we write $X = \langle \mathbf{c}_1, \ldots, \mathbf{c}_n\rangle_{\mathbb{F}_q}$.*

Let $\mathbf{G}$ be the generator matrix of a non-degenerate $[n, k]$ rank-metric code $\mathcal{C}$ over $\mathbb{F}_{q^m}/\mathbb{F}_q$. The $\mathbb{F}_q$ space generated by the columns of $\mathbf{G}$ is an $[n, k]$ q-system and in [9], it is shown that this gives a one-to-one correspondence between equivalence classes of q-systems and rank-metric codes. The key point is the geometric interpretation of rank of a codeword which we explain below.

Lemma 1. *For a vector* $\mathbf{c} \in \mathbb{F}_{q^m}^n$, $\mathrm{rank}(\mathbf{c}) = n - \dim_{\mathbb{F}_q} \mathrm{Ker}\, \phi_{\mathbf{c}}$ *where*

$$\phi_{\mathbf{c}} \colon \mathbb{F}_q^n \longrightarrow \mathbb{F}_{q^m} \text{ is given by } \mathbf{a} \longmapsto \mathbf{c} \cdot \mathbf{a}.$$

Proof. It is a straightforward consequence of the rank-nullity theorem. □

Let $\mathcal{C}$ be an $[n, k]$ rank-metric code over $\mathbb{F}_{q^m}/\mathbb{F}_q$. For any codeword $\mathbf{c} \in \mathcal{C}$, $\mathbf{c} = \mathbf{x_c}\mathbf{G}$ for some $\mathbf{x_c} \in \mathbb{F}_{q^m}^k$. Then $\mathbf{c}$ defines a unique hyperplane $H_{\mathbf{c}}$ of $\mathbb{F}_{q^m}^k$ given by the kernel of the map $\psi_{\mathbf{c}}$ defined as follows

$$\begin{aligned} \psi_{\mathbf{c}} \colon \mathbb{F}_{q^m}^k &\longrightarrow \mathbb{F}_{q^m} \\ \mathbf{e} &\longmapsto \mathbf{x_c} \cdot \mathbf{e}. \end{aligned} \tag{1}$$

Conversely, a hyperplane H of $\mathbb{F}_{q^m}^k$ defined by a vector $\mathbf{x} \in \mathbb{F}_{q^m}^k$ defines a codeword $\mathbf{c} = \mathbf{x}\mathbf{G}$.

Lemma 2. *Let* $\mathcal{C}$ *be an* $[n, k]$ *rank-metric code over* $\mathbb{F}_{q^m}/\mathbb{F}_q$ *and let* X *be the* q*-system corresponding to a generator matrix* $\mathbf{G}$ *of* $\mathcal{C}$*. Then for any* $\mathbf{c} \in \mathcal{C}$,

$$\mathrm{rank}(\mathbf{c}) = n - \dim_{\mathbb{F}_q}(X \cap H_{\mathbf{c}}), where \tag{2}$$

$H_{\mathbf{c}} = \mathrm{Ker}\ \psi_{\mathbf{c}}$ *is considered as an* $\mathbb{F}_q$*-subspace of* $\mathbb{F}_{q^m}^k$ *of dimension* $m(k-1)$.

Proof. For $\mathbf{e} \in (X \cap H_{\mathbf{c}})$, $\mathbf{x_c} \cdot \mathbf{e} = 0$ and $\mathbf{e} = \mathbf{G}\mathbf{a}^T$ for some $\mathbf{a} \in \mathbb{F}_q^n$. Note that $\mathbf{a}$ is unique since $\mathcal{C}$ is non-degenerate. This implies that $\mathbf{c} \cdot \mathbf{a} = 0$. Therefore, $\mathbf{e} \in X \cap H_{\mathbf{c}}$ corresponds to a unique element $\mathbf{a} \in \mathrm{Ker}\ \phi_{\mathbf{c}}$. Also, the same argument backward shows that any element of $\mathrm{Ker}\, \phi_{\mathbf{c}}$ corresponds to a unique element of $X \cap H_{\mathbf{c}}$. Hence $|X \cap H_{\mathbf{c}}| = |\mathrm{Ker}\, \phi_{\mathbf{c}}|$ and the result follows from Lemma 1. □

The geometric approach helps to completely classify the constant weight rank-metric codes. We recall the classification below.

Definition 2. *[9, Definition 11] Let* X *be the* $\mathbb{F}_q$*-vector space* $\mathbb{F}_{q^m}^k$ *of dimension* mk*. A Hadamard rank-metric code* $H_1(q, m, k)$ *is an* $[mk, k, m]$ *linear code with generator matrix where the columns are made of* $\mathbb{F}_q$*-basis of* X*.*

Theorem 1. *[9, Theorem 12] Let* $\mathcal{C}$ *be an* $[n, k, d]$*-non-degenerate constant weight linear rank metric code over* $\mathbb{F}_{q^m}/\mathbb{F}_q$*. Then*

(a) for $k = 1$, $\mathcal{C} = \langle (a_1, \ldots, a_n) \rangle_{\mathbb{F}_{q^m}}$ *where* $\mathrm{rank}(a_1, \ldots, a_n) = d$*, and*
(b) for $k > 1$, $\mathcal{C}$ *is a* $[mk, k, m]$*-Hadamard rank-metric code* $H_1(q, m, k)$*.*

2.2 t-Spreads

Spreads are widely studied objects in finite geometry [1,6,2]. Here we collect the relevant basic notions regarding spreads.

Definition 3. *A* t*-spread is a pair* (V, Σ) *where* V *is a vector space of dimension* n *over* $\mathbb{F}_q$ *and* Σ *is a set of subspaces of* V *of dimension* t *such that* $\bigcup_{S \in \Sigma} S = V$ *and for all* $S_1 \neq S_2 \in \Sigma$, $S_1 \cap S_2 = \{\mathbf{0}\}$*. If the ambient space* V *is clear from the context, we simply write* Σ *to denote the* t*-spread.*

Whenever we simply write "spread", we mean that (V,Σ) is a t-spread where $t=(\dim_{\mathbb{F}_q} V)/2$. The space V has a t-spread if and only if t-divides n [10].

Two spreads $\Sigma_1,\ \Sigma_2$ of V are called *equivalent* if there exists a collineation α of $\Gamma L(V)$ such that $\Sigma_1^\alpha=\Sigma_2$.

Suppose $n=lt$. Any r-dimensional $\mathbb{F}_{q^t}$-subspace of $\mathbb{F}_{q^t}^l$ is an rt-dimensional $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^t}^l$. Now, let $\mathcal{H}$ be the set of all one dimensional subspaces of $\mathbb{F}_{q^t}^l$ and thus $\#\mathcal{H}=\frac{q^n-1}{q^t-1}$. These 1-dimensional $\mathbb{F}_{q^t}$-subspaces are considered as t-dimensional $\mathbb{F}_q$-subspaces of $\mathbb{F}_{q^t}^l\simeq\mathbb{F}_q^{tl}=\mathbb{F}_q^n$ and they intersect trivially. Hence, up to isomorphism, $\mathcal{H}$ is a set of $\frac{q^n-1}{q^t-1}$ non-intersecting t-dimensional subspaces of $\mathbb{F}_q^n$.

Definition 4 (Desarguesian t-spread). *Let $\mathcal{D}_{l,t,q}$ be the t-spread of $\mathbb{F}_{q^t}^l$, where $\mathcal{D}_{l,t,q}$ is the set of t-dimensional subspaces of $\mathbb{F}_{q^t}^l$ defined by the 1-dimensional $\mathbb{F}_{q^t}$-subspaces of $\mathbb{F}_{q^t}^l$. A t-spread Σ is called Desarguesian if Σ is equivalent to $\mathcal{D}_{l,t,q}$ for some r, t and q.*

Definition 5. *Let V be an $\mathbb{F}_q$-vector space. Let (V,Σ) be a t-spread in V. For a subspace W of V, the projection of Σ onto W is the set $\Sigma_W=\{S\cap W : S\in\Sigma\}\backslash\{\{\mathbf{0}\}\}$. The pair (W,Σ_W) is called a subsystem of (V,S).*

For any arbitrary subspace $W\subset V$, the projection Σ_W is not necessarily a t'-spread. In fact the dimension of the elements of Σ_W can be distinct.

Definition 6. *Let (V,Σ) be a t-spread in V and let $(W,\Sigma_W))$ be a subsystem of (V,S). If (W,S_W) is itself t'-spread for some $t'\le t$, then it is called a t'-subspread induced by (V,S) on W. If a t'-subspread is a spread, then it is simply called a subspread.*

These generalize the notion of subspreads of [1, Definition 4.4.1] to general t.

3 Antipodal two-weight rank-metric codes and t-spreads

In this section, we discuss properties of the *antipodal* two-weight rank-metric codes and t-spreads and establish a relation between these two objects.

3.1 Antipodal two-weight (ATW) rank-metric codes

An $[n,k,d]$ ATW rank-metric code over $\mathbb{F}_{q^m}/\mathbb{F}_q$, we must have $n\le m$ because it has a codeword of full rank n. We give a first description of the form of the generator matrix of an ATW rank-metric code. As we progress, we will refine this form.

Lemma 3. *Let $\mathcal{C}$ be a non-degenerate $[n,k,d]$ ATW rank-metric code over $\mathbb{F}_{q^m}/\mathbb{F}_q$. Then $\mathcal{C}$ is equivalent to a rank-metric code with a generator matrix of the form*

$$\mathbf{G}=\begin{pmatrix}\mathbf{c}_1 & \mathbf{c}_2\\ \mathbf{A} & \mathbf{0}\end{pmatrix},\ \textit{where}$$

(i) $\mathbf{c}_1 \in \mathbb{F}_{q^m}^{n-r}$, $\mathbf{c}_2 \in \mathbb{F}_{q^m}^{r}$ *for some positive integer* r *and* $\operatorname{rank}(\mathbf{c}_1|\mathbf{c}_2) = n$.

(ii) $\mathbf{A} \in \mathbb{F}_{q^m}^{(k-1)\times(n-r)}$ *is a generator matrix of a non-degenerate constant weight rank-metric code.*

Proof. Since $\mathcal{C}$ is antipodal, there is a codeword $\mathbf{c}$ of full rank n. After row reductions, we can have a generator matrix of the following form $\mathbf{G}' = \begin{pmatrix} \mathbf{c}' & \alpha \\ \mathbf{A}' & \mathbf{0}_{(k-1)\times 1} \end{pmatrix}$, where $\mathbf{c}' \in \mathbb{F}_{q^m}^{n-1}$, $\alpha \in \mathbb{F}_{q^m}^{\times}$ such that $\mathbf{c} = (\mathbf{c}'|\alpha)$ and $\mathbf{A}' \in \mathbb{F}_{q^m}^{(k-1)\times(n-1)}$. Since the subcode $\mathcal{C}' \subset \mathcal{C}$ generated by $\left[\mathbf{A}'|\mathbf{0}_{(k-1)\times 1}\right]$ can't have a codeword of rank n, then it must be a constant rank weight code with minimum distance d. By a proper invertible matrix $\mathbf{M} \in \mathbb{F}_q^{n\times n}$, we can write $\left[\mathbf{A}'|\mathbf{0}_{(k-1)\times 1}\right]\mathbf{M} = \left[\mathbf{A}|\mathbf{0}_{(k-1)\times r}\right]$, where $1 \le r \le n-1$ and $\mathbf{A} \in \mathbb{F}_{q^m}^{(k-1)\times(n-r)}$ is a generator matrix of a non-degenerate constant weight rank-metric code. Then $\mathbf{G} := \mathbf{G}'\mathbf{M}$ has the desired form and the codeword $(\mathbf{c}_1|\mathbf{c}_2) = (\mathbf{c}'|\alpha)\mathbf{M}$ has rank n since M is invertible. □

It is clear that any two-weight rank-metric code cannot be of dimension 1. Now, following Theorem 1, we can say that the matrix $\mathbf{A}$ in the decomposition of G in Lemma 3 generates either a code of dimension 1 or a Hadamard rank-metric code. And the later case leads to the following result.

Theorem 2. *There is no* $[n, k, d]$ *ATW rank-metric code over* $\mathbb{F}_{q^m}/\mathbb{F}_q$ *of dimension* $k \ge 3$.

Proof. Let $\mathcal{C}$ be an $[n, k, d]$ ATW rank-metric code with of dimension $k \ge 3$ and generator matrix $\mathbf{G}$. By Lemma 3, we can assume that $\mathbf{G} = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 \\ \mathbf{A} & \mathbf{0} \end{pmatrix}$, where $\mathbf{A}$ is a generator matrix of a non-degenerate constant weight rank-metric code of dimension $k-1 \ge 2$ and minimum distance d. Also, following Theorem 1, $\mathbf{A}$ is a generator matrix of a Hadamard code $H_1(q, m, k-1)$. But that implies $m = d < n \le m$ which is a contradiction. □

So any ATW rank-metric codes are only of dimension 2. The next lemma gives a restriction on the distance of the code.

Lemma 4. *For any* $[n, 2, d]$ *ATW rank-metric code, the minimum distance* d *is at least* $n/2$.

Proof. Let $\mathcal{C}$ be an $[n, 2, d]$ ATW rank-metric code. Following Lemma 3, $\mathcal{C}$ is equivalent to a code $\mathcal{C}'$ with a generator matrix $\mathbf{G} = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 \\ \mathbf{c}_3 & \mathbf{0} \end{pmatrix}$, where $\mathbf{c}_1, \mathbf{c}_3 \in \mathbb{F}_{q^m}^{d}$, $\mathbf{c}_2 \in \mathbb{F}_{q^m}^{n-d}$, $\operatorname{rank}(\mathbf{c}_3) = d$ and $\operatorname{rank}(\mathbf{c}_1|\mathbf{c}_2) = n$. After row reduction, $\mathbf{G}$ can be transformed into $\mathbf{G}' = \begin{pmatrix} \mathbf{c}_1' & \mathbf{c}_2 \\ \mathbf{c}_3 & \mathbf{0} \end{pmatrix}$, where the first entry of $\mathbf{c}_1'$ is zero and two rows have rank d. Now $\operatorname{rank}(\mathbf{c}_1|\mathbf{c}_2) = n$ implies that $\mathbf{c}_2$ has full rank $n-d$. Since $\operatorname{rank}(\mathbf{c}_2) \le \operatorname{rank}(\mathbf{c}_1'|\mathbf{c}_2) = d$, then $d \ge n-d$ or $d \ge n/2$. □

The following example gives an ATW rank-metric code with $d = n/2$. Later we show that any codes of the same parameters much be equivalent to this one.

Example 1. Let $\mathbb{F}_{q^d} = \langle \alpha_1, \dots, \alpha_d \rangle_{\mathbb{F}_q}$ be an extension of degree d over $\mathbb{F}_q$. Then the matrix $\mathbf{G} = \begin{pmatrix} 0 & 0 & \dots & 0 & \alpha_1 & \alpha_2 & \dots & \alpha_d \\ \alpha_1 & \alpha_2 & \dots & \alpha_d & 0 & 0 & \dots & 0 \end{pmatrix}$ generates a non-degenerate ATW $[n = 2d, 2, d]$ rank-metric code $\mathcal{C}$ over $\mathbb{F}/\mathbb{F}_q$, where $\mathbb{F}$ is any proper extension of $\mathbb{F}_{q^d}$. For the proof, we know that all codewords of $\mathcal{C}$ are of the form $(a_1, a_2)\mathbf{G}$, where either $a_2 a_1^{-1} \in \mathbb{F}_q$ or $a_2 a_1^{-1} \notin \mathbb{F}_q$. The first case gives us a codeword of rank d whereas the second case gives us a codeword of rank n.

3.2 Some properties of t-spreads

Here we prove some properties of t-spreads that will be useful in later sections.

Lemma 5. *Let (Σ, V) be a t-spread of V where V is a vector space over $\mathbb{F}_q$ of dimension $n = tl$. For any integer $1 < r < l$, if $\{S_1, S_2, \dots, S_r\}$ are distinct subspaces in Σ such that $\dim_{\mathbb{F}_q} S_1 + \dots + S_r = tr$, then there is an element S_{r+1} in Σ such that $\dim_{\mathbb{F}_q} S_1 + \dots + S_{r+1} = t(r+1)$.*

Proof. Suppose that for any $S_{r+1} \neq S_i$, $i = 1, \dots, r$, $\dim_{\mathbb{F}_q} S_1 + \dots + S_{r+1} < t(r+1)$. Hence $S_{r+1} \cap (S_1 + \dots + S_r)$ contains a non-zero vector. Since the elements of a t-spread pairwise intersect trivially, we get $\#(S_1 + \dots + S_r) \setminus \{\mathbf{0}\} \geq \frac{q^n - 1}{q^t - 1} - r$. Therefore $q^{tr} - 1 \geq \frac{q^n - 1}{q^t - 1} - r$ and thus $q^{t(r+1)} - q^{tr} \geq q^n - 1 - (r-1)(q^t - 1)$. Therefore $q^n - q^{tr} \geq q^n - 1 - (r-1)(q^t - 1)$ (As $q^n \geq q^{t(r+1)}$, for $r + 1 \leq l$). By simplifying, we get $q^{tr} - 1 \leq (r-1)q^t$ and so $q^{tr} \leq rq^t$, which leads to a contradiction and hence the Lemma is proved. $\square$

Theorem 3. *Suppose that V is a vector space of dimension n over $\mathbb{F}_q$ and let $n = tl$. Let Σ be a t-spread of V. Then there are $S_1, \dots, S_l$ in Σ such that $V = S_1 \oplus S_2 \oplus \dots \oplus S_l$.*

Proof. Choose S_1 and S_2 as any elements of Σ. By the definition of t-spreads, $S_1 + S_2$ is a direct sum. The previous lemma says that we can increase it to $S_1 \oplus S_2 \oplus S_3$ and applying the lemma repetitively, we get the result. $\square$

3.3 Relation between ATW codes and t-spreads

We now give a relation between $(n-d)$-spreads and ATW rank-metric codes with $d \geq n/2$. The following lemma is obtained by using the same method as used in classifying constant rank weight codes in [9].

Lemma 6. *Let $\mathcal{C}$ be an $[n, 2, d]$ ATW rank-metric code over $\mathbb{F}_{q^m}/\mathbb{F}_q$ with a generator matrix $\mathbf{G}$ and let X be the q-system corresponding to $\mathbf{G}$. If $\mathcal{H}$ be the set of all hyperplanes of $\mathbb{F}_{q^m}^2$, then we can partition $\mathcal{H}$ into $\mathcal{H}_1 \sqcup \mathcal{H}_2$ where*

$$\mathcal{H}_1 = \{H \in \mathcal{H} \colon \dim_{\mathbb{F}_q} H \cap X = n - d\}, \mathcal{H}_2 = \{H \in \mathcal{H} \colon \dim_{\mathbb{F}_q} H \cap X = 0\}. \quad (3)$$

Proof. Let $\mathbf{c} \in \mathcal{C}$ be a nonzero codeword and let $H_{\mathbf{c}}$ be the corresponding hyperplane. Then Lemma 2 implies that $\mathbb{F}_q$-dimension of $H \cap X$ is $n-d$ if $\mathrm{rank}(\mathbf{c}) = d$ and 0 othwerwise. Since any hyperplane of $\mathbb{F}_{q^m}^2$ corresponds to some codewords, we get the desired partition of $\mathcal{H}$ by considering $\mathcal{H}_1$ (resp. $\mathcal{H}_2$) to be the set of hyperplanes corresponding to codewords of rank d (resp. n). $\square$

Proposition 1. *Let $\mathcal{C}$ be an $[n, 2, d]$ ATW rank-metric code over $\mathbb{F}_{q^m}/\mathbb{F}_q$ with a generator matrix $\mathbf{G}$ and let X be the q-system corresponding to $\mathbf{G}$. Let $\mathcal{H}$ be the set of all hyperplanes of $\mathbb{F}_{q^m}^2$ and $\mathcal{H}_1 = \{H \in \mathcal{H} \colon \dim_{\mathbb{F}_q} H \cap X = n-d\}$. Then $|\mathcal{H}_1| = \frac{q^n-1}{q^{n-d}-1}$ and thus the set $\Sigma = \{H \cap X : H \in \mathcal{H}_1\}$ defines an $(n-d)$-spread (X, Σ) of X.*

Proof. Lemma 6 implies $\mathcal{H} = \mathcal{H}_1 \sqcup \mathcal{H}_2$ where $\mathcal{H}_2$ as in equation (3). Note that the hyperplanes have pairwise trivial intersection. Therefore we can partition the set $\mathbb{F}_{q^m}^2 \backslash \{\mathbf{0}\}$ as

$$\mathbb{F}_{q^m}^2 \backslash \{\mathbf{0}\} = \left(\bigsqcup_{H \in \mathcal{H}_1} H \backslash \{\mathbf{0}\} \right) \bigsqcup \left(\bigsqcup_{H \in \mathcal{H}_2} H \backslash \{\mathbf{0}\} \right), \tag{4}$$

where $H \backslash \{\mathbf{0}\}$ is the set of non-zero elements of H. Define the valuation v on $\mathbb{F}_{q^m}^2$ by $v(\mathbf{a}) = \begin{cases} 1 & \text{if } \mathbf{a} \in X, \\ 0 & \text{otherwise,} \end{cases}$ and by abusing notation, $v(S) = \sum_{\mathbf{a} \in S} v(\mathbf{a})$ for any subset S of $\mathbb{F}_{q^m}^2$. Since all the unions in equation (4) are disjoint, then

$$v(\mathbb{F}_{q^m}^2 \backslash \{\mathbf{0}\}) = \sum_{H \in \mathcal{H}_1} v\left(H \backslash \{\mathbf{0}\}\right) + \sum_{H \in \mathcal{H}_2} v\left(H \backslash \{\mathbf{0}\}\right),$$

Now $v(H \backslash \{\mathbf{0}\}) = q^{n-d} - 1$ for $H \in \mathcal{H}_1$, and $v(H \backslash \{\mathbf{0}\}) = 0$ if $H \in \mathcal{H}_2$. Furthermore $v(\mathbb{F}_{q^m}^2 \backslash \{\mathbf{0}\}) = v(X \backslash \{\mathbf{0}\})$. Therefore, $q^n - 1 = |\mathcal{H}_1|(q^{n-d} - 1)$. The pairwise trivial intersection of elements of Σ implies that Σ is an $(n-d)$-spread. $\square$

Corollary 1. *If $\mathcal{C}$ is an $[n, 2, d]$ ATW rank-metric code, then $n-d$ divides n.*

Proof. Proposition 1 implies that a q-system X has an $(n-d)$-spread Σ such that $|\Sigma| = \frac{q^n-1}{q^{n-d}-1}$ and thus $n-d$ divides n. $\square$

Theorem 4. *Let $\mathcal{C}$ be an $[n, 2, d]$ ATW rank-metric code. Suppose that $n = l(n-d)$, then $\mathcal{C}$ is equivalent to a rank-metric code with generator matrix $\mathbf{G} = [\mathbf{G}_1 | \dots | \mathbf{G}_l]$, where the columns of each $\mathbf{G}_i$'s belong to a hyperplane H_i and H_i's are pairwise distinct.*

Proof. Proposition 1 says that X admits an $(n-d)$-spread. Then we choose the columns of $\mathbf{G}$ from the decomposition in Theorem 3, where each block $\mathbf{G}_i$ consists of a $\mathbb{F}_q$-basis of $H_i \cap X$ for $H_i \in \mathcal{H}_1$. $\square$

We have shown that ATW rank-metric codes define t-spreads in the q-system X. The following theorem shows that these t-spreads satisfy a certain property.

Theorem 5. *Let $\mathcal{C}$ be an $[n, 2, d]$ rank-metric code over $\mathbb{F}_{q^m}/\mathbb{F}_q$ generated by $\mathbf{G}$ with the corresponding q-system X. Let Δ be a Desarguesian m-spread on $\mathbb{F}_{q^m}^2$. The following assertions are equivalent.*

(i) *$\mathcal{C}$ is an ATW rank-metric code.*
(ii) *$(\mathbb{F}_{q^m}^2, \Delta)$ induces an $(n-d)$-spread (X, Δ_X) of X.*

Proof.(i) $\Rightarrow$ (ii): Since $\mathcal{C}$ is an ATW rank-metric code, following Proposition 1 we know that $\Delta_X = \{X \cap H : H \in \mathcal{H}_1\}$ is an $(n-d)$-spread of X. Also, from Definition 6 it follows that (X, Δ_X) is a $(n-d)$-subspread of $(\mathbb{F}_{q^m}^2, \Delta)$.

(ii) $\Rightarrow$ (i): Suppose that (X, Δ_X) is an $(n-d)$-subspread of $(\mathbb{F}_{q^m}^2, \Delta)$. Let $\mathcal{H}$ be the set of all hyperplanes in $\mathbb{F}_{q^m}^2$. Then from Definition 4 and Definition 6, it follows that an element of $\mathcal{H}$ intersects X is an $\mathbb{F}_q$-space of dimension either $n-d$ or 0. From Lemma 2, we have that for any $H \in \mathcal{H}$, $H = H_{\mathbf{c}}$ for some $\mathbf{c} \in \mathcal{C}$ and conversely, any $\mathbf{c} \in \mathcal{C}$ defines a hyperplane $H_{\mathbf{c}}$ (two linearly dependent codewords define the same hyperplane). Then Equation (2) in Lemma 2 implies that the only possible rank for any non-zero codewords are d, n. □

In the following result, we give a condition for any general $(n-d)$-spreads of $\mathbb{F}_q^n$ to induce an $[n, 2, d]$ ATW rank-metric codes.

Corollary 2. *Let n, m, d be positive integers with $d \le n \le m$ and let Σ be an $(n-d)$-spread of $\mathbb{F}_q^n$. Let $X \subseteq \mathbb{F}_{q^m}^2$ be a q-system and suppose that $\mathbf{G}$ is a $(2 \times n)$-matrix whose columns consist of an $\mathbb{F}_q$-basis of X. If for any $S \in \Sigma$, $\mathbf{G}S^T = \{\mathbf{G}\mathbf{x}^T : \mathbf{x} \in S\}$ is contained in a hyperplane H of $\mathbb{F}_{q^m}^2$ and each different S correspond to different H, then $\mathbf{G}$ generates an ATW rank-metric code.*

4 Constructions

In Example 1 of Section 3.1 we have seen a construction of ATW rank-metric codes. We construct ATW rank-metric codes for more general parameters using suitable MRD codes.

Theorem 6. *Let d, l, m, n be positive integers such that $d \le n \le m$, $(n-d)|m$, and $n = l(n-d)$. Suppose $\mathcal{C}$ be a non-degenerate $[l, 2, l-1]$ MRD code over $\mathbb{F}_{q^m}/\mathbb{F}_{q^{n-d}}$ with a generator matrix $\mathbf{G}$. Fix a basis $\{a_1, \ldots, a_{n-d}\}$ of $\mathbb{F}_{q^{n-d}}$ over $\mathbb{F}_q$ and let $\tilde{\mathbf{G}}$ be the $2 \times l(n-d)$ block matrix $(\tilde{\mathbf{G}}_1| \ldots |\tilde{\mathbf{G}}_l)$ where the $(n-d)$ columns of $\tilde{\mathbf{G}}_i$ are given by $\{a_j \mathbf{G}_i, j = 1, 2, \ldots, n-d\}$ with $\mathbf{G}_i$ being the i-th column of G. Then $\tilde{G}$ generates an $[n, 2, d]$ ATW rank-metric code $\tilde{\mathcal{C}}$ over $\mathbb{F}_{q^m}/\mathbb{F}_q$.*

Proof. We claim that if the codeword $c = (c_1, c_2)\mathbf{G}$ of $\mathcal{C}$ has rank weight l (respectively, $l-1$), then the codeword $\tilde{c} = (c_1, c_2)\tilde{\mathbf{G}}$ of $\tilde{\mathcal{C}}$ has rank weight $l(n-d)$ (respectively, $(l-1)(n-d)$). It is easy to see that if the claim is proved then $\tilde{\mathcal{C}}$ is an ATW $[n, 2, d]$ code over $\mathbb{F}_{q^m}/\mathbb{F}_q$.

Proof of the claim: Let X (resp. $\tilde{X}$) be the q-system corresponding to the generator matrix $\mathbf{G}$ (resp. $\tilde{\mathbf{G}}$). It is crucial to note that the $\mathbb{F}_q$-space $\tilde{X}$ and the $\mathbb{F}_{q^{n-d}}$-space X are same as set. Now let the codeword $c = (c_1, c_2)\mathbf{G}$ of $\mathcal{C}$ has rank weight l. Thus from equation (1), we get $\dim_{\mathbb{F}_{q^{n-d}}} H_c \cap X = 0$ and hence, $\dim_{\mathbb{F}_q} H_c \cap \tilde{X} = 0$. Therefore, $rank(\tilde{c}) = n - \dim_{\mathbb{F}_q} H_c \cap \tilde{X} = n$. Similarly, for the case when $rank(c) = l - 1$ we have $\dim_{\mathbb{F}_{q^{n-d}}} H_c \cap X = 1$ (follows from equation (1)). Then $\dim_{\mathbb{F}_q} H_c \cap \tilde{X} = \dim_{\mathbb{F}_q} H_c \cap X = n - d$ and therefore, $rank(\tilde{c}) = n - (n - d) = d$. □

The above theorem says that if X is a q^{n-d}-system corresponding to an $[l, 2, l-1]$ MRD code over $\mathbb{F}_{q^m}/\mathbb{F}_{q^{n-d}}$, then X is a q-system corresponding to an ATW $[n, 2, d]$ code $\tilde{\mathcal{C}}$ over $\mathbb{F}_{q^m}/\mathbb{F}_q$. We call the code $\tilde{\mathcal{C}}$ the *ATW rank-metric code induced* by the MRD rank-metric code $\mathcal{C}$.

Corollary 3. *Let $d, l, m, n,$ be positive integers such that $d \leq n \leq m$, $n = l(n-d)$, and $(n-d)|m$. Let X be a q-system corresponding to an $[n, 2, d]$ ATW code $\tilde{\mathcal{C}}$ over $\mathbb{F}_{q^m}/\mathbb{F}_q$. Then $\tilde{\mathcal{C}}$ is induced by an $[l, 2, l-1]$ MRD code $\mathcal{C}$ over $\mathbb{F}_{q^m}/\mathbb{F}_{q^{n-d}}$ if and only if X is also an $\mathbb{F}_{q^{n-d}}$-space.*

It is natural to ask if all ATW rank-metric codes are induced by MRD codes as shown in the previous construction. The answer is affirmative when $d = n/2$.

Theorem 7. *Let $\mathcal{C}$ be an $[n, 2, d = n/2]$ rank-metric code over $\mathbb{F}_{q^m}/\mathbb{F}_q$. Then $\mathcal{C}$ is an ATW rank-metric code if and only if d divides m and $\mathcal{C}$ is equivalent to a code with generator matrix*

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 & \alpha_2 & \dots & \alpha_d \\ 1 & \alpha_2 & \dots & \alpha_d & 0 & 0 & \dots & 0 \end{pmatrix}, \text{ such that } \langle 1, \alpha_2, \dots, \alpha_d \rangle_{\mathbb{F}_q} = \mathbb{F}_{q^d}.$$

Proof. Following Theorem 4, we know that $\mathcal{C}$ has a generator matrix of the form $\mathbf{G} = [\mathbf{G}_1|\mathbf{G}_2]$ where the columns of $\mathbf{G}_1$ (resp. $\mathbf{G}_2$) belong to a hyperplane H_1 (resp. H_2). For $i = 1, 2$, let $\mathbf{x}_i \in \mathbb{F}_{q^m}^2$ such that $H_i = \text{Ker } \psi_{\mathbf{c}_i}$ where $\mathbf{c}_i = \mathbf{x}_i\mathbf{G}$ as in Equation (1). Then $\mathcal{C}$ has another generator matrix $\mathbf{G}' = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \mathbf{G} = \begin{pmatrix} 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & \mathbf{0} \end{pmatrix}$, where $\mathbf{e}_1$ and $\mathbf{e}_2$ both have rank d.

Without loss of generality we can assume that $\mathbf{e}_1 = (1, \alpha_2 \dots, \alpha_d)$ and $\mathbf{e}_2 = (1, \beta_2, \dots, \beta_d)$ where $\alpha_i, \beta_i \in \mathbb{F}_{q^m}^{\times}$. Take the codeword $(\mathbf{e}_2|\mathbf{e}_1)$, because the first and second part both contain 1, then $0 < \text{rank}(\mathbf{e}_2|\mathbf{e}_1) < n$ and therefore, $\text{rank}(\mathbf{e}_2|\mathbf{e}_1) = d$. Thus the β_i's are $\mathbb{F}_q$-linear combination of the $\alpha_i' s$ and 1. Hence, $\mathcal{C}$ is equivalent to a code with generator matrix $\overline{\mathbf{G}}$ of the form $\overline{\mathbf{G}} = \begin{pmatrix} \mathbf{0} & \mathbf{e}_1 \\ \mathbf{e}_1 & \mathbf{0} \end{pmatrix}$. For $i = 2, \dots, d$, let $\mathbf{c}^{(i)} = (\mathbf{e}_1|\alpha_i\mathbf{e}_1)$. Because the first and second part of $\mathbf{c}^{(i)}$ contain α_i, then $0 < \text{rank}(\mathbf{c}^{(i)}) < n$ and thus $\text{rank}(\mathbf{c}^{(i)}) = d$. This implies that $\alpha_i\alpha_j \in \langle 1, \alpha_2, \dots, \alpha_d \rangle_{\mathbb{F}_q}$ for all $2 \leq i, j \leq d$. So the space $\langle 1, \alpha_2, \dots, \alpha_d \rangle_{\mathbb{F}_q}$ is in fact a subfield, say K, of $\mathbb{F}_{q^m}$ with $[K : \mathbb{F}_q] = d$ and hence $d \mid m$. Therefore we get the desired form of the generator matrix. For the converse, it follows from the arguments in Example 1 that $\mathbf{G}_1$ generates a non-degenerate ATW rank-metric codes. □

For $d = n/2$, we showed that the ATW rank-metric code essentially corresponds to a Desarguesian spread $(\mathbb{F}_{q^{n/2}}, \Delta)$.

Corollary 4. *Let $(\mathbb{F}_{q^m}^2, \Delta)$ be a Desarguesian spread. Let $X \subseteq \mathbb{F}_{q^m}^2$ be an $\mathbb{F}_q$-space of dimension n and suppose that $(\mathbb{F}_{q^m}^2, \Delta)$ induces a subspread, i.e. $n/2$-subspread, (X, Δ_X) of X, then (X, Δ_X) is a Desarguesian spread.*

Proof. Since (X, Δ_X) is a subspread, then by definition, the elements of Δ_X has dimension $n/2$. Following Theorem 5, X defines an $[n, 2, n/2]$ ATW rank-metric code. Theorem 7 implies that each element S of Δ_X is an $\mathbb{F}_{q^{n/2}}$ vector space. So the induced subspread (X, Δ_X) is a Desarguesian spread of X. □

Theorem 6 gives a construction of $[n, 2, d]$ ATW rank-metric codes over $\mathbb{F}_{q^m}/\mathbb{F}_q$ from suitable MRD codes over $\mathbb{F}_{q^m}/\mathbb{F}_{q^{n-d}}$. In fact, for the case $d = n/2$ this is the only way of constructing ATW rank-metric codes as proved in Theorem 7. So it is natural to ask if this happens for the case when $d > n/2$? In other words, are there any $[n, 2, d]$ ATW rank-metric code which are not induced by MRD codes? Equivalently, in terms of t-spreads, one may ask if a t-subspread of a Desarguesian spread is again a Desarguesian t-spread.

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